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Stabilizers of collineation groups of smooth stable planes

by Richard Bödi

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
e-mail: richard.boedi@uni-tuebingen.de

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ABSTRACT

Let $\mathcal{S}$ be a smooth stable plane of dimension n (see Definition 1.2) and let Δ be a closed subgroup of the collineation group of $\mathcal{S}$ which fixes some point p. We derive some results on the group-theoretical structure of Δ, e.g. that Δ is a linear Lie group (Theorem 3.7). As a by-product this shows that no (affine or projective) Moulton plane can be turned into a smooth plane. If Δ fixes some flag, then any Levi subgroup Ψ of Δ is a compact group and Δ is contained in the flag stabilizer of the classical Moufang plane of dimension n (Corollary 3.1 and Theorem 3.7). Let Δ fix three concurrent lines through the point p. If $\mathcal{S}$ is one of the classical projective planes over the reals, the complex numbers, the quaternions, or the Cayley numbers, then the dimension of Δ is $d_{\text{class}} = 3, 6, 15$, or 38, respectively. We show that for a smooth stable (projective) plane $\mathcal{S}$ of dimension $2l$ either $\mathcal{S}$ is an almost projective translation plane (classical projective plane) or that $\dim \Delta \leq d_{\text{class}} - l$ holds (Theorems 4.1 and 4.2).

1. INTRODUCTION

In this paper we continue the investigation of collineation groups of smooth stable planes focusing on stabilizers of points and flags. The concept of a smooth stable plane has been introduced in [2]. Throughout we will use the definitions and the notation of [2] and [3]. We recall the most basic definitions of [2].

Definition 1.1. A *linear space* is a triple $\mathcal{S} = (P, \mathcal{L}, \mathcal{F})$ of sets P, $\mathcal{L}$ and $\mathcal{F}$, where P denotes the *set of points*, $\mathcal{L}$ is the *set of lines* and $\mathcal{F} \subseteq P \times \mathcal{L}$ is the *set of flags*, such that for every pair of distinct points p,q there is exactly one *joining line*

$L \in \mathcal{L}$, i.e. $(p, L), (q, L) \in \mathcal{F}$. If $(p, L) \in \mathcal{F}$, we will say that p and L are incident, or that p lies on L, or that L passes through p.

Definition 1.2. A *stable plane* $\mathcal{S}$ is a linear space $(P, \mathcal{L}, \mathcal{F})$ which satisfies the following three axioms:

(S1) There are Hausdorff topologies on both P and $\mathcal{L}$ that are neither discrete nor anti-discrete such that the join map $\vee$ and the intersection map $\wedge$ are continuous. Moreover, the domain $\mathcal{O}$ of the intersection map is an open subset of $\mathcal{L} \times \mathcal{L}$.

(S2) The topology on P is locally compact and has positive finite covering dimension.

(S3) $\mathcal{S}$ contains a quadrangle.

A *smooth stable plane* $\mathcal{S}$ is a stable plane $(P, \mathcal{L}, \mathcal{F})$ such that P and $\mathcal{L}$ are smooth manifolds and such that the join and intersection mappings are smooth on their (respective) domains. If the underlying linear space of $\mathcal{S}$ is a projective plane, then we refer to $\mathcal{S}$ as a *smooth projective plane.*

Let $\mathcal{S} = (P, \mathcal{L}, \mathcal{F})$ always be a smooth stable plane of finite dimension $n = 2^k = 2l$, where $1 \leq k \leq 4$ holds according to Löwen [17], see Salzmann ([25], 7.12) in the case of smooth projective planes. Let Δ denote a connected closed subgroup of the collineation group $\Gamma = \mathrm{Aut}(\mathcal{S})$ of $\mathcal{S}$ which fixes some point $p \in P$. The group Γ of continuous automorphisms of $\mathcal{S}$ (and hence Δ as well) is a Lie transformation group (with respect to the compact-open topology) on both the set P of points and the set $\mathcal{L}$ of lines, see ([3], 2.4). Moreover, every continuous collineation of $\mathcal{S}$ is in fact a smooth map on P and on $\mathcal{L}$, ([3], 2.3), cf. [4]. We denote by $\mathcal{A}_2\mathbb{F}$ the affine translation plane over the (possibly non-commutative) field $\mathbb{F}$ and we write $\mathcal{P}_2\mathbb{F}$ for its projective closure.

One of the most beautiful results in the theory of smooth projective planes is due to J. Otte [22] and [23]. His result will be essential in our study of stabilizers of three concurrent lines (see Section 4).

Otte's theorem 1.3. *Every smooth projective translation plane is isomorphic to one of the four classical Moufang planes* $\mathcal{P}_2\mathbb{F}$, *where* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

2. AUXILIARY RESULTS

For the investigation of the group Δ we utilize results of H. Hähl about collineation groups of locally compact affine translation planes ([10], 4.2): let $\mathcal{A} = (A, \mathcal{L}, \mathcal{F})$ be a locally compact affine translation plane of dimension $n = 2l$. We choose some point o in A as well as three distinct lines $W, S, X \in \mathcal{L}_o$ through o. Fixing a 'unit point' e in $X \setminus \{o\}$, the affine translation plane $\mathcal{A}$ is coordinatized by some quasifield Q whose additive group $(Q, +)$ is isomorphic to $\mathbb{R}^l$. Hence, the kernel of the quasifield Q contains the real numbers as a subfield. In particular, the group $(Q, +)$ can be viewed as an l-dimensional real vector space. In this setting, the set A of points can be written as $A = Q \times Q \cong \mathbb{R}^{2l}$, the origin o has coordinates (0,0), and we have $W = Q \times$

$\{0\}$, $S = \{0\} \times Q$, and $X = \operatorname{diag}(Q \times Q)$. The automorphism group Σ of $\mathcal{A}$ is a semi-direct product $\Sigma = \Sigma_o \ltimes T$, where $T \cong \mathbb{R}^{2l}$ is the group of translations and Σ_o is the stabilizer of the origin o. Moreover, the stabilizer $\Sigma_{W,S}$ can be expressed in terms of $\mathbb{R}$-linear mappings of the real vector space Q, namely

$$\Sigma_{W,S} \leq \{(B, C) : Q^2 \to Q^2 : (x, y) \mapsto (Bx, Cy) \mid B, C \in \mathrm{GL}(Q)\}.$$

Since we have $X = \operatorname{diag}(Q \times Q)$, the stabilizer of the three lines S, W and X can be written as

$$\Sigma_{W,S,X} \leq \{(B, B) : Q^2 \to Q^2 : (x, y) \mapsto (Bx, By) \mid B \in \mathrm{GL}(Q)\}.$$

We need a theorem by Hähl (see [12], 2.1 or [27], 81.8), which turns out to be a very effective tool.

Theorem 2.1. *Let $\mathcal{A}$ be an affine locally compact translation plane of dimension $n = 2l$ and let W, S and X be three different lines of $\mathcal{A}$ through some point o. Let Φ be the connected component of the collineation group Σ of $\mathcal{A}$.*

(a) *The group $M_2 := \{(A, B) \in \Phi \mid |\det A| = |\det B| = 1\} \leq \mathrm{SO}_l\mathbb{R} \times \mathrm{SO}_l\mathbb{R}$ is the largest compact subgroup of the stabilizer $\Theta_2 := \Phi_{W,S}$ and $\dim \Theta_2/_{M_2} \leq 2$.*

(b) *The group $M_3 := \{(A, A) \in \Phi \mid |\det A| = 1\} \leq \mathrm{SO}_l\mathbb{R}$ is the largest compact subgroup of the stabilizer $\Theta_3 := \Phi_{W,S,X}$ and $\dim \Theta_3/_{M_3} \leq 1$. Moreover, there is a closed one-parameter subgroup P in Θ_3 such that $\Theta_3^1 = M_3 \times P$.*

The next two lemmas provide useful formulas for the dimension $\dim \Delta$ of Δ which we will use throughout without further mentioning. Whenever we write $\dim X$ for some topological space X we always mean its covering dimension. Note that for a locally compact group G the covering dimension coincides with the small and the large inductive dimension. The same is of course true for the spaces P, $\mathcal{L}$ and $\mathcal{F}$ of the smooth stable plane $\mathcal{S}$, since these spaces are smooth manifolds (cp. [2], 2.14). In fact, it is even true for (topological) stable planes, see Löwen [17]. We will need this fact in the proof of Lemma 2.3. For more information on this topic see ([27], §5 and §92). The following lemma is due to Halder [13], see also ([27], 96.15).

Lemma 2.2. *Let G be a locally compact Lindelöf group acting on a separable metric space M. For every element $\alpha \in M$ with $\dim \alpha^M < \infty$ we have $\dim G = \dim \alpha^G + \dim G_\alpha$, where G_α denotes the stabilizer of G at α and α^G is the orbit of α under G.*

By [2], the tangent spaces $\mathrm{T}_q P$ at each point $q \in P$ together with the tangent spreads $\mathcal{S}_q = \{\mathrm{T}_q L \mid L \in \mathcal{L}_q\}$ constitute locally compact affine translation planes $\mathcal{A}_q$. The stabilizer Γ_q of some point $q \in P$ induces a (continuous) action on the tangent translation plane $\mathcal{A}_q$ via the *derivation mapping*

$$D_q : \Gamma_q \to \mathrm{Aut}(\mathcal{A}_q)_o : \gamma \mapsto D\gamma(q),$$

where $\mathrm{Aut}(\mathcal{A}_q)_o$ denotes the stabilizer of $\mathrm{Aut}(\mathcal{A}_q)$ at the origin o. By ([3], 3.3), the map D_q is continuous.

Lemma 2.3. *For any point $q \in P$ we have* $\dim \Gamma_q = \dim \ker \mathrm{D}_q + \dim \mathrm{D}_q\Gamma_q$ *and* $\dim \mathrm{D}_q\Gamma_q \leq \dim \mathrm{Aut}(\mathcal{A}_q)_o$. *Moreover, we have* $\ker \mathrm{D}_q = \Gamma_{[q,q]}$.

Proof. According to Theorem 3.3 of [3] the stabilizer $\Lambda := \Gamma_q$ acts continuously on the linear Lie group $\Sigma_o := \mathrm{Aut}(\mathcal{A}_q)_o$ via composition $\Lambda \times \Sigma_o \to \Sigma_o : (\lambda, \sigma) \mapsto \mathrm{D}_q(\lambda)\sigma$. By Lemma 2.2 this implies $\dim \Lambda - \dim \Lambda_{\mathbb{I}} = \dim \mathbb{I}^{\Lambda} = \dim \mathrm{D}_q\Lambda \leq \dim \Sigma_o$, where the inequality follows from the monotonicity of the small inductive dimension. From $\Lambda_{\mathbb{I}} = \ker \mathrm{D}_q$ we conclude that $\dim \Lambda - \dim \ker \mathrm{D}_q = \dim \mathrm{D}_q\Lambda$, and hence the first part of the lemma is proved. The second claim is proved in ([3], 4.18). $\square$

We continue with a remark on the structure of the elation groups $\Gamma_{[p,p]}$. In the case of compact connected topological translation planes such groups are isomorphic to $\mathbb{R}^n$, where n is the dimension of the plane. If there exist elations with different axes in $\Gamma_{[p,p]}$, then for arbitrary compact connected topological planes it is known that $\Gamma_{[p,p]}$ is abelian and that its connected component $\Gamma^1_{[p,p]}$ is isomorphic to some vector group $\mathbb{R}^k$, see Salzmann ([26], §1(1)), H. Lüneburg ([18], I, 4.9), and M. Lüneburg ([19], I, Z1 and Z2). If there do not exist elations with different axes, little is known in the topological case. For smooth stable planes, however, the situation is much the same as for translation planes: it can be shown that $\Gamma^1_{[p,p]}$ is always isomorphic to some vector group $\mathbb{R}^m$, see Theorem 4.17 of [3]. If $m = n$, then $\Gamma_{[p,p]}$ is connected and hence we have $\Gamma_{[p,p]} \cong \mathbb{R}^n$. Moreover, for n-dimensional smooth stable planes, the existence of an n-dimensional group $\Gamma_{[p,p]}$ of elations has similar consequences as for n-dimensional projective planes.

Theorem 2.4. *If there is a point p such that* $\dim \Gamma_{[p,p]} = n$, *then $\mathcal{S}$ is an almost projective translation plane, i.e. there is a compact connected (not necessarily smooth) projective translation plane $\mathcal{P} = (P', \mathcal{L}')$ with translation axis W such that $P = P' \setminus C$, where $C \neq W$ is some closed subset of W, and $\mathcal{L} \setminus \{W\} \subseteq \mathcal{L}' \subseteq \mathcal{L}$.*

Proof. By the preceding remark we have $\Gamma_{[p,p]} \cong \mathbb{R}^n$, and Stroppel ([28], 3.6) yields that $\mathcal{S}$ is an almost projective topological translation plane. Finally, we have $C \neq W$ for $\mathcal{S}$ has a compact line by Corollary 4.7 of [3] (note that there are no compact lines in a locally compact affine plane). $\square$

As an immediate consequence of the last result Otte's theorem together with Lemma 2.3 provides the following corollary.

Corollary 2.5. *Either $\mathcal{S}$ is an almost projective translation plane or the inequality* $\dim \Delta < n + \dim \mathrm{D}_p\Delta$ *holds. If, in addition, $\mathcal{S}$ is a smooth projective plane, then either $\mathcal{S}$ is a classical Moufang plane $\mathcal{P}_2\mathbb{F}$ or we have* $\dim \Delta < n + \dim \mathrm{D}_p\Delta$.

The torus rank (the dimension of a maximal torus subgroup) of the stabilizer Θ_3 (or equivalently, of M_3) of a 2^k-dimensional locally compact affine translation plane $\mathcal{A}$ cannot reach the exponent k (cp. [27], 55.37), i.e. following lemma holds.

Lemma 2.6. *We have* $\operatorname{rk} \Theta_3 < k$.

Proof. By M. Lüneburg ([19], II, Satz 2) (see also [27], 55.37), the torus rank r of the automorphism group Γ of the projective closure $\mathcal{P}$ of $\mathcal{A}$ cannot exceed the exponent k of the dimension 2^k of $\mathcal{A}$. Moreover, the group Γ contains three different commuting reflections if $r = k \geq 2$. The group Θ_3, considered as a group of collineations of $\mathcal{P}$ fixing the line L_∞ at infinity, fixes the degenerate quadrangle defined by the lines W, S, X, and L_∞. Thus the reflections in question must have the common center $0 = W \wedge S$ which contradicts a result of ([27], 55.35). Hence we have $r < k$ if $k \geq 2$. It remains to study the case $k = 1$. Then, the plane $\mathcal{P}$ is isomorphic to the real projective plane (see Saltzmann [25], 7.24), and the group Θ_3 is a subgroup of the additive group of the real numbers. In particular, the group Θ_3 has no torus subgroup, whence $\operatorname{rk} \Theta_3 = 0$ follows. □

3. GENERAL RESULTS

We are going to recall some facts about semi-simple subgroups of a Lie group. Let Ψ be a connected semi-simple subgroup of the connected Lie group Δ. Since Ψ need not be closed in Δ, we consider Ψ with respect to its Lie topology τ_{Lie} which turns Ψ into a Lie group. A basis of τ_{Lie} is given by the arc components of open sets in the induced topology, see Gleason and Palais ([8], 3.2 and 4.2). Note that this topology may differ from the induced topology of Δ. In any case, the Lie topology τ_{Lie} is finer than the induced topology. Since Ψ and $(\Psi, \tau_{\text{Lie}})$ do have the same one-parameter subgroups ([8], 3.2(5)), the topology τ_{Lie} is not the discrete topology. In fact, this is true for every almost simple factor Φ of Ψ. Every non-trivial normal subgroup of an almost simple Lie group is contained in its center and thus is countable. Since the connected component of the identity of a topological group is a normal subgroup, and because Φ has a non-trivial one-parameter subgroup, this implies that $(\Phi, \tau_{\text{Lie}})$ is connected as well. Being a product of almost simple factors, we infer that $(\Psi, \tau_{\text{Lie}})$ is still connected.

By a Levi subgroup Ψ of a Lie group we mean a maximal semi-simple analytic subgroup. In particular, such a subgroup Ψ is connected by the definition of a analytic group (see Hochschild [14], p. 114). Note, however, that this differs from the definition of a Levi subgroup given in Onishchik and Vinberg ([21], p. 287), where connectedness is not assumed. We start with the examination of flag stabilizers.

Proposition 3.1. *If the connected Lie group* Δ *of collineations of a smooth stable* *plane S fixes some flag (p, L), then every Levi subgroup Ψ of Δ is a compact group and the derivation maps $\mathrm{D}_p : \Psi \to \mathrm{GL}(\mathrm{T}_p P)$ and $\mathrm{D}_L : \Psi \to \mathrm{GL}(\mathrm{T}_L \mathcal{L})$ are injective.*

Proof. We have mentioned already that the Lie topology τ_{Lie} of Ψ is finer than the induced topology of Ψ in Δ. Hence, the homomorphism $\mathrm{D}_p : (\Psi, \tau_{\mathrm{Lie}}) \to \mathrm{GL}(\mathrm{T}_p P)$ is still continuous. We will simply write $\mathrm{D}_p\Psi$ for the image of $(\Psi, \tau_{\mathrm{Lie}})$ under the map D_p. Since Ψ is semi-simple and connected, its image $\mathrm{D}_p\Psi$ is closed in $\mathrm{GL}(\mathrm{T}_p P)$ by van Est [7] or Gotô [9], and $\mathrm{D}_p\Psi$ acts completely reducibly on the tangent space $\mathrm{T}_p P$, cp. ([21], Chapter 5, §2, Corollary 2). Moreover, the group $\mathrm{D}_p\Psi$ leaves the l-dimensional subspace $\mathrm{T}_p L \in \mathcal{S}_p$ invariant. Thus there is a $\mathrm{D}_p\Psi$-invariant complement W of $\mathrm{T}_p L$ in $\mathrm{T}_p P$. Let $\mathfrak{g}$ denote the (real) Lie algebra of $\mathrm{D}_p\Psi$.

Assume that the group $\mathrm{D}_p\Psi$ is not compact. Then there is a non-trivial subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is $\mathbb{R}$-diagonalizable, i.e. all operators $\mathrm{ad}\,x$ for $x \in \mathfrak{a}$ can be expressed by diagonal matrices with respect to a suitable basis of $\mathfrak{g}$, see ([21], Chapter 5, §4, p.268–270). Moreover, the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is ρ'-diagonalizable for every representation $\rho' : \mathfrak{g} \to \mathrm{gl}(V)$, i.e. every element $\rho'(x)$ for $x \in \mathfrak{a}$ is represented by a diagonal matrix with respect to some fixed basis of V, see ([21], Chapter 5, p. 277). We denote by ρ the representation of $\mathrm{D}_p\Psi$ on W and we put $\rho' : \mathfrak{g} \to \mathrm{gl}(W)$ as the associated representation of the Lie algebra $\mathfrak{g}$ of $\mathrm{D}_p\Psi$. Since $\mathfrak{a} \neq \{0\}$ is ρ'-diagonalizable, there is a one-dimensional subspace $\mathbb{R}(v_1, \ldots, v_l)$ of $\mathfrak{a}$ which is mapped by the exponential mapping onto the one-parameter subgroup $P = \{(\exp tv_1, \ldots, \exp tv_l) \mid t \in \mathbb{R}\}$ of $\rho(\mathrm{D}_p\Psi)$. In particular, the group P is not relatively compact, because $(v_1, \ldots, v_l) \neq 0$. By Weil's lemma, we infer that P is a closed subgroup of $\mathrm{GL}(W)$ which fixes the one-dimensional subspaces $W_i := (0, \ldots, 0, \exp tv_i, 0, \ldots, 0)$. Since P acts on $\mathcal{S}_p$, the group P fixes the unique spread element S of $\mathcal{S}_p$ that contains W_1, say. Thus, the one-parameter group P fixes the two lines $\mathrm{T}_p L$ and S. Because $\mathrm{D}_p\Psi$ is semi-simple and leaves $\mathrm{T}_p L$ invariant, we have $\det \rho(g)\,|_{\mathrm{T}_p L} = \det \rho(g)\,|_S = 1$ for every $g \in P$, and consequently P is relatively compact by Theorem 2.1, contrary to our assumption. Hence, the group $\mathrm{D}_p\Psi$ is compact.

In order to verify that $\mathrm{D}_p\,|_\Psi$ is an injection, we recall that no almost simple factor of Ψ lies in the kernel of D_p according to ([3], 4.17 and 4.18). Thus $\ker(\mathrm{D}_p\,|_\Psi)$ is contained in the center $\mathrm{Z}(\Psi)$ of Ψ. Since $\mathrm{Z}(\Psi)$ is discrete with respect to τ_{Lie}, we conclude that Ψ is a covering group of $\mathrm{D}_p\Psi \cong \Psi/\ker(\mathrm{D}_p\,|_\Psi)$. Recalling that Ψ is semi-simple, this implies that $(\Psi, \tau_{\mathrm{Lie}})$ is a compact group too, and we infer that the center $\mathrm{Z}(\Psi)$ is finite, see Bredon ([6], 0.6.10) or ([27], 94.29). Because the induced topology on Ψ is coarser than τ_{Lie}, this shows that Ψ is compact as well and hence is closed in Δ. By ([3], Theorem 4.17), this shows that $\ker(\mathrm{D}_p\,|_\Psi) \cap \mathrm{Z}(\Psi) = \{\mathbb{1}\}$ and so $\mathrm{D}_p\,|_\Psi$ is injective. Finally, the map $\mathrm{D}_L : \Psi \to \mathrm{GL}(\mathrm{T}_L \mathcal{L})$ is injective by Theorem 3.3 of [3]. $\square$

The proof of the last proposition also verifies the compactness of Levi complements of certain stabilizers in the case of compact projective translation planes:

Corollary 3.2. *Let $\mathcal{P}$ be a compact connected translation plane with translation axis L_∞, and let W be some line of $\mathcal{P}$ through the origin o. Let Υ be the stabilizer of the collineation group of $\mathcal{P}$ that fixes both lines W and L_∞ as well as the origin o. Then every Levi complement of Υ is compact.*

Corollary 3.3. *The center of any semi-simple subgroup of Δ does not contain an elation.*

Proof. Let Z be the center of a semi-simple subgroup Ψ of Δ. Assume that $\mathbb{1} \neq \zeta \in Z$ is an elation with center c and axis A. Since ζ fixes no point outside of A according to ([3], Lemma 4.5), the group Ψ leaves invariant the axis A. Analogously, the group Ψ fixes the center c (otherwise we would have a fixed point outside A again). Hence the semi-simple group Ψ fixes the flag (c, A). By Proposition 3.1 this implies that Ψ is compact and thus has a finite center. In particular, the elation ζ has a finite order which is a contradiction to ([3], Corollary 4.10). $\square$

Corollary 3.4. *Let Δ fix some flag (p, L) and let Ψ be a Levi subgroup of Δ. Then $\Upsilon := \Psi \ltimes \Gamma^1_{[p,p]}$ is a linear group and Ψ is a maximal compact subgroup of Υ.*

Proof. By Proposition 3.1, the Levi subgroup Ψ is compact and linear. By Hochschild ([14], Chapter 18, 3.1 and 4.2), the group Υ is linear if its radical $\Gamma^1_{[p,p]}$ is simply connected, which is true by Theorem 4.17 of [3]. Since Ψ is compact and $\Gamma^1_{[p,p]}$ contains no non-trivial compact subgroup, we conclude that $\Psi \cap \Gamma^1_{[p,p]} = \mathbb{1}$. Hence Υ is a semi-direct product of Ψ and $\Gamma^1_{[p,p]}$. $\square$

We turn to stabilizers of points.

Proposition 3.5. *Every Levi subgroup Ψ of Δ is a linear group.*

Proof. Choose a Levi complement Ψ of Δ. We consider the continuous homomorphism $\mathrm{D}_p : \Psi \to \mathrm{GL}(\mathrm{T}_p P)$. Assume that Ψ is not a linear group. Then the center $\mathrm{Z}(\Psi)$ is infinite and the kernel $\ker \mathrm{D}_p \subseteq \mathrm{Z}(\Psi)$ is non-trivial. By Theorem 4.18 and Corollary 4.9 of [3] we have $\ker \mathrm{D}_p = \Delta_{[p,p]} = \bigcup_{A \in \mathcal{L}_p} \Delta_{[p,A]}$. Let $\zeta \in \ker \mathrm{D}_p \backslash \{\mathbb{1}\}$ with $L \in \mathcal{L}_p$ as its axis and choose some point $q \in L \backslash \{p\}$. If $q^\Delta \subseteq p \vee q$, then Δ fixes the flag $(p, p \vee q)$ and the claim follows from Proposition 3.1. If, on the other hand, we have $q^\Delta \not\subseteq p \vee q$, then $r = q^\delta \notin p \vee q$ for some $\delta \in \Delta$ and $(r \vee q)^\zeta = (q^\delta \vee q)^\zeta = q^{\delta\zeta} \vee q^\zeta = q^{\zeta\delta} \vee q^\zeta = q^\delta \vee q = r \vee q$, i.e. the line $K = r \vee q$ is fixed by ζ. But this is impossible, since $K \notin \mathcal{L}_p$ and $\zeta \in \Delta_{[p,L]}$. This contradiction shows that Ψ is a linear group. $\square$

Applied to smooth projective planes, this proposition shows that a non-classical compact projective Moulton plane cannot be turned into a smooth projective plane. For an introduction into Moulton planes the reader is referred to Salzmann ([25], §4) and ([27], §34). The first proof of this fact is due to Betten [1]. However, in contrast to our proof, his proof is not based merely on the collineation group of a Moulton plane.

Corollary 3.6. *Let $\mathcal{P}$ be a compact connected projective Moulton plane which is not isomorphic to the real projective plane. Then there are no smooth structures on the set of points and the set of lines such that $\mathcal{P}$ becomes a smooth projective plane.*

Proof. The collineation group Γ of $\mathcal{P}$ is isomorphic to some product $\mathbb{R}\tilde{\Omega}$, where $\tilde{\Omega}$ is the universal covering group of $\mathrm{PSL}_2\mathbb{R}$. The group $\tilde{\Omega}$ is not a linear group, see e.g. ([21], Table 10), and fixes an antiflag (p_0, L_0). Thus $\mathcal{P}$ cannot be turned into a smooth projective plane according to Corollary 3.5. □

Remark. By removing the line L_0 of the non-classical Moufang plane $\mathcal{P}$ we obtain a locally compact affine plane $\mathcal{A}$ having the same collineation group as $\mathcal{P}$. Arguing as in the proof of Corollary 3.6 we obtain an example of a locally compact connected affine plane which cannot be turned into a smooth affine plane.

Now we are going to generalize Proposition 3.5.

Theorem 3.7. *Δ is a linear group.*

Proof. Let Ψ denote a Levi subgroup of Δ and let A be the solvable radical of Δ. By Proposition 3.5, the group Ψ is linear. Thus it is sufficient to show that A is a linear group, see Hochschild ([14], Chapter XVIII, Theorem 4.2). We have shown this in ([3], 4.19). Hence the theorem is proved. □

In the next theorem we prove that every Levi subgroup of a flag stabilizer of a smooth stable plane is classical, i.e. it is isomorphic to some subgroup of one of the flag stabilizers of one of the classical planes $\mathcal{P}_2\mathbb{F}$.

Theorem 3.8. *Let Δ be a connected subgroup of the stabilizer of some flag (p, L). If $\dim \mathcal{S} \leq 4$, then Δ is solvable. If $\dim \mathcal{S} = 8$ or 16, then every Levi subgroup Ψ of Δ is locally isomorphic to some closed subgroup of $(\mathrm{Spin}_3\mathbb{R})^3$ or $\mathrm{Spin}_8\mathbb{R}$, respectively.*

Proof. Let Ψ be a Levi subgroup of Δ. By Proposition 3.1, the group Ψ is compact and the restrictions of D_p and D_L to Ψ are injective. Since Δ fixes the line L, the image $\mathrm{D}_p\Delta$ leaves the corresponding tangent space T_pL invariant. Being semi-simple, the group Δ acts completely reducibly on $\mathrm{T}_pP \cong \mathbb{R}^{2l}$ and thus leaves invariant some complement W of T_pL in T_pP. Every quasi-simple factor Σ of $\mathrm{D}_p\Psi$ acts almost effectively on at least one of the subspaces T_pL and W.

Thus, if $\dim \mathcal{S} \leq 4$, we conclude that $\Sigma = \mathbb{1}$, since a compact quasi-simple group does not have any linear representation of dimension less than 3. Hence Ψ is solvable if $\dim \mathcal{S} \leq 4$. It remains to prove the second assertion. If $\mathrm{D}_p\Psi$ acts almost faithfully (i.e. with a discrete kernel) on the subspace W, then Ψ is locally isomorphic to some subgroup of $\mathrm{SO}(W) \cong \mathrm{SO}_l\mathbb{R}$ (recall that Ψ is compact). Thus, the assertion follows, because $\mathrm{SO}_4\mathbb{R}$ is locally isomorphic to $\mathrm{Spin}_3\mathbb{R} \times \mathrm{Spin}_3\mathbb{R}$.

Hence, let us assume that the kernel N of the action of $\mathrm{D}_p\Psi$ on W is not discrete. Then N is a non-trivial semi-simple group having a quasi-simple factor N_ι. If $W \in \mathcal{S}_p$, then N consists of shears with axis W. The group of shears with axis W is commutative, because $\mathcal{A}_p$ is a translation plane. This contradicts the fact that N is quasi-simple. Hence we may assume that W is not contained in $\mathcal{S}_p$. Then there are points w, w' in W with $o \vee w \neq o \vee w'$. Since both lines $o \vee w$ and $o \vee w'$ of $\mathcal{A}_p$ intersect W in a vector subspace of positive dimension, the group N fixes a non-degenerate quadrangle. Thus, the substructure $\mathcal{E}$ of fixed points and fixed lines of N forms a subplane of $\mathcal{A}_p$. Since W has dimension l, the affine plane $\mathcal{E}$ is a Baer subplane of $\mathcal{A}_p$. Since N is a normal subgroup of $\mathrm{D}_p\Psi$, the latter group leaves $\mathcal{E}$ invariant. Let K denote the kernel of the action of $\mathrm{D}_p\Psi$ on $\mathcal{E}$ and let $\Omega = \mathrm{D}_p\Psi/_K$. According to ([27], 83.18 and 83.22), the kernel K is a subgroup of $\mathrm{Spin}_3\mathbb{R}$ and thus $\dim K \leq 3$ holds. The line T_pL is fixed by $\mathrm{D}_p\Psi$ and therefore is an inner line of $\mathcal{E}$, i.e. it is incident with at least two different points of $\mathcal{E}$. In fact, the points of $\mathcal{E}$ lying on T_pL form a $l/2$-dimensional linear subspace of the point set E of $\mathcal{E}$. Set $k = l/2$. Being quasi-simple and compact, the group Ω acts fully reducibly on E. This gives rise to a faithful representation $\Omega \hookrightarrow \mathrm{SO}_k\mathbb{R} \times \mathrm{SO}_k\mathbb{R}$. This forces Ω to be trivial, if $k = 2$. In the remaining case $k = 4$ we infer that $\mathrm{D}_p\Psi$ is isomorphic to a product of m groups of type $\mathrm{Spin}_3\mathbb{R}$. Due to the remark before Lemma 2.6 we have $m \leq 3$. This completes the proof of the theorem. □

4. THE DIMENSION OF THE STABILIZER OF THREE CONCURRENT LINES

We are now going to consider the subgroup Λ of Δ which fixes three lines K, L, and M of the smooth stable plane $\mathcal{S}$ through the point p. If $\mathcal{S}$ is one of the classical Moufang planes $\mathcal{P}_2\mathbb{F}$ of dimension n, we denote by $\mathrm{d}_{\mathrm{class}}(n)$ the dimension of Λ. The values of $\mathrm{d}_{\mathrm{class}}(n)$ are as follows:

n	2	4	8	16
$\mathrm{d}_{\mathrm{class}}(n)$	3	6	18	38

Our aim is to derive upper bounds for the dimensions of Λ in terms of $\mathrm{d}_{\mathrm{class}}(n)$. More precisely, we prove the following theorems.

Theorem 4.1. *Let $\mathcal{S}$ be a smooth stable plane of dimension $n = 2l$ with Γ as its group of continuous collineations. Let Λ be a closed subgroup of Γ that fixes three distinct lines through some point p. Then either the plane $\mathcal{S}$ is an almost projective translation plane or we have* $\dim \Lambda \leq \mathrm{d}_{\mathrm{class}}(n) - l$.

Theorem 4.2. *Let $\mathcal{P}$ be a smooth projective plane of dimension $n = 2l$ with Γ as its group of continuous collineations. Let Λ be a closed subgroup of Γ that fixes three distinct lines through some point p. Then either the plane $\mathcal{P}$ is classical or we have* $\dim \Lambda \leq \mathrm{d}_{\text{class}}(n) - l$.

We denote by A the group of continuous collineations of the tangent translation plane $\mathcal{A}_p$. Again, we put $\Theta_3 := A_{\mathrm{T}_pK, \mathrm{T}_pL, \mathrm{T}_pM}$. Since the group Λ fixes the three lines K, L, M, the image $\mathrm{D}_p\Lambda$ fixes the three distinct lines $\mathrm{T}_pK, \mathrm{T}_pL$, and T_pM of $\mathcal{A}_p$. Thus we have $\mathrm{D}_p : \Lambda \to \Theta_3$. By Theorem 2.1, the group Θ_3 has a compact subgroup $M_3 \leq \mathrm{SO}_l\mathbb{R}$ of codimension at most 1. We will verify the two theorems above for every possible n separately.

Theorem 4.3. *Let* $\dim \mathcal{S} = 2$. *Then either $\mathcal{S}$ is an almost projective translation plane or* $\dim \Lambda \leq 2$. *If $\mathcal{S}$ is a smooth projective plane, then either $\mathcal{S} \cong \mathcal{P}_2\mathbb{R}$ or* $\dim \Lambda \leq 2$.

Proof. Since $\dim \Theta_3 \leq 1$ for $l = 1$, we infer that $\dim \mathrm{D}_p\Lambda \leq 1$. From Corollary 2.5 we conclude that either $\mathcal{S}$ is an almost projective translation plane (if $\mathcal{S}$ is a smooth projective plane, then $\mathcal{S}$ is even classical by Otte's theorem), or that $\dim \Lambda < 2 + \dim \mathrm{D}_p\Lambda \leq 3$ holds. □

Before we move on to higher dimensional smooth stable planes, we will formulate a rather elementary lemma which nevertheless will play an important role in our proofs.

Lemma 4.4. *Let (p, N) be some flag of $\mathcal{S}$ which is fixed by Δ. If $\alpha \in \Delta_{[p]} \backslash \Delta_{[p,p]}$ centralizes the elation group $\Delta_{[p,N]}$, then $\Delta_{[p,N]} = \mathbb{1}$.*

Proof. Let $\delta \in \Delta_{[p,N]} \backslash \{\mathbb{1}\}$. Then α possesses an axis A (which of course does not pass through p) by ([3], Corollary 4.8). Moreover, there is a point $q \in A \backslash (A \wedge N)$ and a point $r \in (p \vee q) \backslash \{q\}$ such that $r^\delta = q$. Thus we have $r^\alpha = r^{\delta\alpha\delta^{-1}} = q^{\alpha\delta^{-1}} = q^{\delta^{-1}} = r$ which implies that $\alpha = \mathbb{1}$, see Löwen ([15], 3.4) and ([16], 3.4). This contradiction shows that $\Delta_{[p,N]}$ is the trivial group. □

In view of Corollary 2.5 we may suppose that $\dim \Delta_{[p,p]} < n$, if $\mathcal{S}$ is not an almost projective translation plane. If $\dim \Delta_{[p,N]} = 0$ for some line $N \in \mathcal{L}_p$, then $\dim \Delta_{[p,p]} \leq l$ by ([27], 61.11). Since any classical compact projective plane has a collineation group that is transitive on (degenerated) quadrangles, we obtain in this case by using Lemma 2.3

$$\dim \Delta \leq \dim \Delta_{[p,p]} + \dim \Theta_3 \leq l + \mathrm{d}_{\text{class}}(n) - 2l = \mathrm{d}_{\text{class}}(n) - l.$$

Hence we may always assume in our proofs that, say, $1 \leq \dim \Delta_{[p,K]} < l$ holds. Note that this gives another proof of Theorem 4.3, because $\dim \Delta_{[p,K]} = 1 = \dim \Delta_{[p,L]}$ implies that $\dim \Delta_{[p,p]} = 2$ and thus Theorem 2.4 proves Theorem 4.3.

Theorem 4.5. *Let* $\dim \mathcal{S} = 4$. *Then either* $\mathcal{S}$ *is an almost projective translation plane or* $\dim \Lambda \leq 4$. *If* $\mathcal{S}$ *is a smooth projective plane, either* $\mathcal{S} \cong \mathcal{P}_2\mathbb{C}$ *or* $\dim \Lambda \leq 4$.

Proof. According to the remark before Theorem 4.3, the image $\mathrm{D}_p\Lambda$ is contained in the group Θ_3. By Theorem 2.1, the group Θ_3 has a compact subgroup $M_3 \leq \mathrm{SO}_2\mathbb{R}$ and a closed one-parameter subgroup P such that $\Theta_3 = M_3 \times P \leq \mathbb{C}^\times$. Furthermore we have either $\dim \mathrm{D}_p\Lambda \leq 1$ (which gives $\dim \Lambda < 4 + 1 = 5$ by Corollary 2.5) or $\mathrm{D}_p\Lambda \cong \mathbb{C}^\times$. In the latter case, the group $\mathrm{D}_p\Lambda$ has a central involution $\omega = -\mathbb{1}$. Clearly, this involution fixes every element of the tangent spread $\mathcal{S}_p$. Hence, the involution ω is a reflection with center 0. Choose an element $\alpha \in \mathrm{D}_p^{-1}(\omega)$. Since $\omega = -\mathbb{1}$ fixes every line T_pH for $H \in \mathcal{L}_p$, the collineation α fixes every line H through p. Thus, α is a central collineation with center p. According to Corollary 4.8 of [3], the existence of elations ensures that α has an axis A. The axis A does not pass through p, for otherwise we would have $\mathrm{D}_p\alpha \,|\, \mathrm{T}_pA = \mathbb{1}$, which contradicts the fact that $\mathrm{D}_p\alpha = \omega = -\mathbb{1}$. By our general assumption we have $\dim \Delta_{[p,K]} = 1$ and thus $\Delta^1_{[p,K]} \cong \mathbb{R}$ holds according to Theorem 4.17 of [3]. Since the connected group Λ fixes the line K, it acts on $\Delta^1_{[p,K]}$ via conjugation. Because Λ is connected, this action gives rise to a continuous homomorphism $\Lambda \to (\mathrm{GL}_1\mathbb{R})^1 \cong \mathbb{R}_{>0}$. The homology α is an involution, too, since we have $\alpha^2 \in \ker \mathrm{D}_p = \Delta_{[p,p]}$ and $\Delta_{[p,p]} \cap \Delta_{[p,A]} = \{\mathbb{1}\}$ by ([3], Lemma 4.5). Hence α acts trivially on $\Delta^1_{[p,K]}$. Since Λ is connected, the involution α acts also trivially on $\Delta_{[p,K]}$. Now Lemma 4.4 shows that $\Delta_{[p,K]} = \mathbb{1}$. This contradiction shows that $\dim \mathrm{D}_p\Lambda \leq 1$ and the theorem is proved. $\square$

Before we study the 8-dimensional case, we need two more standard lemmas on Lie groups.

Lemma 4.6. *Let G be a Lie group and let H be a connected solvable Lie subgroup of G. Then the closure $\bar{H}$ of H in G is a connected solvable Lie subgroup of G. If $H \neq \bar{H}$, then* $\dim H < \dim \bar{H}$.

Proof. By Hochschild ([14], XVI, Theorem 2), there is an abelian Lie subgroup A of G such that $\bar{H} = H \cdot A$. Since the direct product $H \times A$ is solvable and because $\bar{H}$ is a quotient of the latter group, we conclude that $\bar{H}$ is solvable, too. If $\dim \bar{H} = \dim H$, then H is an open subgroup of $\bar{H}$. Clearly, the group $\bar{H}$ is connected, since H is connected. Hence, $\bar{H} = H$ follows, because an open subgroup is always closed. $\square$

Lemma 4.7. *The group* $\mathrm{SO}_4\mathbb{R}$ *has no closed subgroup isomorphic to* $\mathrm{SO}_3\mathbb{R} \times \mathrm{SO}_2\mathbb{R}$.

Proof. Making use of the quaternions $\mathbb{H}$, we write $\mathrm{SO}_4\mathbb{R} = \{x \mapsto axb : \mathbb{H} \to \mathbb{H} \mid a, b \in \mathbb{H}'\}$. Any subgroup of $\mathrm{SO}_4\mathbb{R}$ isomorphic to $\mathrm{SO}_3\mathbb{R}$ is conjugate to $\{x \mapsto c^{-1}xc : \mathbb{H} \to \mathbb{H} \mid c \in \mathbb{H}'\}$ which is easily seen to be its own centralizer in $\mathrm{SO}_4\mathbb{R}$. This proves the lemma. $\square$

Theorem 4.8. *Let* $\dim \mathcal{S} = 8$. *Then either* $\mathcal{S}$ *is an almost projective translation plane or* $\dim \Lambda \leq 11$. *If* $\mathcal{S}$ *is a smooth projective plane, either* $\mathcal{S} \cong \mathcal{P}_2\mathbb{H}$ *or* $\dim \Lambda \leq 11$.

Proof. The image $\mathrm{D}_p\Lambda$ is a subgroup of the stabilizer Θ_3, where the latter group is isomorphic to some closed subgroup of $\mathrm{GO}_4\mathbb{R} \cong \mathrm{SO}_4\mathbb{R} \times \mathbb{R}$ according to Theorem 2.1.

Assume that Λ is solvable. Then $\mathrm{D}_p\Lambda$ is a solvable (not necessarily closed) subgroup of Θ_3. The maximal dimension of a closed solvable subgroup of $\mathrm{GO}_4\mathbb{R}$ is three. Hence we have $\dim \mathrm{D}_p\Lambda \leq 3$ by Lemma 4.6, and $\dim \Lambda < 8 + \dim \mathrm{D}_p\Lambda \leq 11$ follows from Corollary 2.5. Thus we may suppose that Λ contains a nontrivial Levi-subgroup Ψ and that $\dim \mathrm{D}_p\Lambda \geq 5$ holds. By Proposition 3.1, the group Ψ is compact and $\mathrm{D}_p \mid_\Psi$ is a closed injection. In particular, this shows that $\mathrm{D}_p\Psi$ is a Levi subgroup of $\mathrm{D}_p\Lambda$. Due to the structure of $\mathrm{GO}_4\mathbb{R}$, this implies that $\Psi \cong \mathrm{D}_p\Psi$ is isomorphic to one of the groups $\mathrm{SU}_2\mathbb{C}$, $\mathrm{SO}_3\mathbb{R}$, $\mathrm{SO}_4\mathbb{R}$.

Let $\sqrt{\mathrm{D}_p\Lambda}$ denote the radical of $\mathrm{D}_p\Lambda$. Then we have $\mathrm{D}_p\Lambda = \mathrm{D}_p\Psi \cdot \sqrt{\mathrm{D}_p\Lambda}$ and $\dim \mathrm{D}_p\Lambda = \dim \mathrm{D}_p\Psi + \dim \sqrt{\mathrm{D}_p\Lambda}$. If $\mathrm{D}_p\Psi$ is locally isomorphic to $\mathrm{SO}_3\mathbb{R}$, then $\dim \mathrm{D}_p\Lambda \geq 5$ implies that $\dim \sqrt{\mathrm{D}_p\Lambda} \geq 2$. Thus we infer that a maximal connected solvable sugroup H of $\mathrm{D}_p\Lambda$ has dimension at least 3. Together with the first part of the proof this shows that in fact $\dim H = 3$. By Lemma 4.6, a connected solvable subgroup of maximal dimension is closed, whence the group $\mathrm{D}_p\Lambda = \mathrm{D}_p\Psi \cdot H$ is closed in Θ_3. Recalling that Θ_3 is a closed connected subgroup of $\mathrm{GO}_4\mathbb{R}$, we moreover get $H \cong \mathbb{T}^2 \times \mathbb{R}$ and $\sqrt{\mathrm{D}_p\Lambda} \cong \mathbb{T} \times \mathbb{R}$. Let ω denote the (unique) involution of $\sqrt{\mathrm{D}_p\Lambda}$. Since $\sqrt{\mathrm{D}_p\Lambda}$ contains exactly one compact subgroup $T \cong \mathbb{T}$, and because $\sqrt{\mathrm{D}_p\Lambda}$ is a normal subgroup of $\mathrm{D}_p\Lambda$, we infer that T lies in the center of $\mathrm{D}_p\Lambda$. In particular, the involution ω is central in $\mathrm{D}_p\Lambda$. By Lemma 4.7, we conclude that $\Psi \neq \mathrm{SO}_3\mathbb{R}$. In other words, we have $\Psi \cong \mathrm{SU}_2\mathbb{C}$ and thus $\mathrm{D}_p\Lambda \cong \mathrm{U}_2\mathbb{C} \times \mathbb{R}$. If $\mathrm{D}_p\Psi \cong \mathrm{SO}_4\mathbb{R}$, we end up with either $\mathrm{D}_p\Lambda \cong \mathrm{SO}_4\mathbb{R}$ or $\mathrm{D}_p\Lambda \cong \mathrm{GO}_4\mathbb{R}$ (and in both cases the group $\mathrm{D}_p\Lambda$ is again a closed subgroup of Θ_3).

Each of the groups $\mathrm{U}_2\mathbb{C} \times \mathbb{R}$, $\mathrm{SO}_4\mathbb{R}$, $\mathrm{GO}_4\mathbb{R}$ has (up to equivalence) a unique 4-dimensional irreducible real representation ρ and possesses a unique central involution ω with $\rho(\omega) = -\mathbb{1}$. Moreover, the involution ω is mapped to $\mathbb{1}$ by any real representation of dimension less than 4. Furthermore, we have shown that Ψ is isomorphic to $\mathrm{SU}_2\mathbb{C}$ or to $\mathrm{SO}_4\mathbb{R}$, and its center $\langle \alpha \rangle \cong \mathbb{Z}_2$ is mapped onto $\langle \omega \rangle$, whence the collineation α is a homology with center p and some axis A, see ([3], Lemma 4.5) as well as Löwen ([15], 3.4), and ([16], 3.4). Since we have $k \leq 3$ by our general assumption, the nontrivial homology $\alpha \in \Delta_{[p]}$ acts trivially on $\Delta^1_{[p,K]} \cong \mathbb{R}^k$. Now we use Lemma 4.4 as in the proof of Theorem 4.5 in order to conclude that $\Delta_{[p,K]}$ is trivial. This contradiction finally gives $\dim \Lambda \leq \dim \Delta_{[p,p]} + \dim \mathrm{D}_p\Lambda \leq 7 + 4 = 11$ and the theorem is proved. $\square$

For 16-dimensional stable planes we will need some information about subgroups of the orthogonal group $SO_8\mathbb{R}$, cp. Hähl ([11], 2.8), and ([27], 95.12).

Lemma 4.9. *Let K be a closed connected subgroup of $SO_8\mathbb{R}$ which does not contain a subgroup isomorphic to $SO_5\mathbb{R}$. Then either K is isomorphic to one of the groups $\mathrm{Spin}_7\mathbb{R}$, $U_4\mathbb{C}$, $SU_4\mathbb{C}$, or $G_{2(-14)}$, or $\dim K \leq 13$ holds. Moreover, we have $K \not\cong U_4\mathbb{C}$, if $\mathrm{rk} K \leq 3$.*

Remark. Since a compact connected topological projective plane does not admit a group $\Gamma \cong SO_5\mathbb{R}$ of collineations, see M. Lüneburg ([19], II, Korollar 1), or ([27], 55.40), we have excluded this case in the lemma above.

Theorem 4.10. *Let $\dim \mathcal{S} = 16$. Then either $\mathcal{S}$ is an almost projective translation plane or $\dim \Lambda \leq 30$. If $\mathcal{S}$ is a smooth projective plane, either $\mathcal{S} \cong \mathcal{P}_2\mathbb{O}$ or $\dim \Lambda \leq 30$.*

Proof. According to Theorem 2.1 the group $D_p\Lambda$ is a subgroup of $GO_8\mathbb{R} \cong SO_8\mathbb{R} \times \mathbb{R}$. No compact connected topological projective plane admits a group $SO_5\mathbb{R}$ of collineations (see M. Lüneburg [19], II, Korollar 1 or [27], 55.40). This rules out the case $D_p\Lambda \cong SO_8\mathbb{R}$. By Lemma 2.6, the group $D_p\Lambda$ has torus rank at most 3. Thus we may apply Lemma 4.9 to the closure H of $D_p\Lambda$ and we obtain that a maximal compact subgroup K of H is either isomorphic to one of the groups $\mathrm{Spin}_7\mathbb{R}$ or $SU_4\mathbb{C}$, or that $\dim K \leq 14$ holds. If $\dim K \leq 14$, then $\dim D_p\Lambda \leq \dim H \leq \dim K + 1 \leq 14 + 1 = 15$ (again by Theorem 2.1), and $\dim \Lambda < 16 + \dim D_p\Lambda \leq 31$ follows by Corollary 2.5. So we may assume that K is isomorphic to one of the quasi-simple groups $\mathrm{Spin}_7\mathbb{R}$ or $SU_4\mathbb{C}$. Now choose some Levi subgroup Ψ of Λ. Since Ψ is compact and because the restriction $D_p \mid_\Psi$ is an injection by Proposition 3.1, we get $\Psi \cong K$ the same way as in the proof of Theorem 4.8. Both groups $\mathrm{Spin}_7\mathbb{R}$ and $SU_4\mathbb{C}$ possess a central involution ω which is mapped onto $-\mathbb{1}$ by the map D_p, and both groups do not have nontrivial real representations of dimension less than 8 (see [29] or [5]). Hence we can apply Lemma 4.4 once again in order to get $\dim \Delta_{[p,p]} \leq 8$. This implies that

$$\dim \Lambda \leq \dim \Delta_{[p,p]} + \dim D_p\Lambda \leq 8 + \dim K + 1 \leq 9 + \dim \mathrm{Spin}_7\mathbb{R} = 9 + 21 = 30,$$

and the theorem is proved. $\square$

REFERENCES

1. Betten, D. – 2-dimensionale differenzierbare projektive Ebenen. Arch. Math. **22**, 304–309 (1971).
2. Bödi, R. – Smooth stable planes. Results Math. **31**, 300–321 (1997).
3. Bödi, R. – Collineations of smooth stable planes (to appear in Forum Math.)
4. Bödi, R. and L. Kramer – Differentiability of continuous homomorphisms between smooth loops. Result. Math. **25**, 13–19 (1994).

5. Bödi, R. and M. Joswig – Tables for an effective enumeration of real representations of quasi-simple Lie groups. Seminar Sophus Lie **3**, 239–253 (1993).
6. Bredon, G.E. – Introduction to Compact Transformation Groups. Academic Press, New York-London (1972).
7. Est, W.A. van – Dense imbeddings of Lie groups. Indag. Math. **13**, 321–328 (1951).
8. Gleason, A.M. and R.S. Palais – On a class of transformation groups. Amer. J. Math. **79**, 631–648 (1957).
9. Gotô, M. – Dense imbedding of topological groups. Proc. Amer. Math. Soc. **4**, 653–655 (1953).
10. Hähl, H. – Automorphismengruppen von lokalkompakten zusammenhängenden Quasikörpern und Translationsebenen. Geom. Dedicata **4**, 305–321 (1975).
11. Hähl, H. – Zur Klassifikation von 8- und 16-dimensionalen lokalkompakten Translationsebenen nach ihren Kollineationsgruppen. Math. Z. **159**, 259–294 (1978).
12. Hähl, H. – Lokalkompakte zusammenhängende Translationsebenen mit großen Sphärenbahnen auf der Translationsachse. Resultate Math. **2**, 62–87 (1979).
13. Halder, H.R. – Dimension der Bahn lokal kompakter Gruppen. Arch. Math. **22**, 302–303 (1971).
14. Hochschild, G. – The structure of Lie groups. Holden-Day, San Francisco, Londen, Amsterdam (1965).
15. Löwen, R. – Vierdimensionale stabile Ebenen. Geom. Dedicta **5**, 239–294 (1976).
16. Löwen, R. – Central collineations and the parallel axiom in stable planes. Geom. Dedicata **10**, 283–315 (1981).
17. Löwen, R. – Topology and dimension of stable planes: On a conjecture of H. Freudenthal. J. Reine Angew. Math. **343**, 108–122 (1983).
18. Lüneburg, H. – Lectures on projective planes. University of Illinois at Chicago Circle (1969).
19. Lüneburg, M. – Involutionen, auflösbare Gruppen und die Klassifikation topologischer Ebenen. Mitt. Math. Sem. Giessen **209**, 1–113 (1992).
20. Montgomery, D. and L. Zippin – Topological transformation groups. Interscience Publ. Inc., New York (1955).
21. Onishchik, A.L and E.B. Vinberg – Lie Groups and Algebraic Groups. Springer, Berlin-Heidelberg-New York (1990).
22. Otte, J. - Differenzierbare Ebenen. Dissertation Kiel (1992).
23. Otte, J. – Smooth Projective Translation Planes. Geom. Dedicata **58**, 203–212 (1995).
24. Richardson, R.W. – Groups acting on the 4-sphere. Illinois J. Math. **5**, 474–485 (1961).
25. Salzmann, H. – Topological Planes. Advances in Math. **2**, 1–60 (1967).
26. Salzmann, H. – Compact 8-dimensional projective planes with large collineation groups. Geom. Dedicata **8**, 139–161 (1979).
27. Salzmann, H., D. Betten, T. Grundhöfer, H. Hähl, R. Löwen and M. Stroppel – Compact Projective Planes. De Gruyter (1995).
28. Stroppel, M. – Quaternion Hermitian planes. Results Math. **23**, 387–397 (1993).
29. Tits, J. – Tabellen zu den einfachen Liegruppen und ihren Darstellungen. Lecture Notes **40**. Springer, Berlin-Heidelberg-New York (1967).

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